On the Complexity of the Circular Chromatic Number

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Abstract

Circular chromatic number, χ_c is a natural generalization of chromatic number. It is known that it is **NP**-hard to determine whether or not an arbitrary graph G satisfies $\chi(G) = \chi_c(G)$. In this paper we prove that this problem is **NP**-hard even if the chromatic number of the graph is known. This answers a question of Xuding Zhu. Also we prove that for all positive integers $k \geq 2$ and $n \geq 3$, for a given graph G with $\chi(G) = n$, it is **NP**-complete to verify if $\chi_c(G) \leq n - \frac{1}{k}$.

1 Introduction

We follow [4] for terminology and notation not defined here, and we consider finite undirected simple graphs. Given a graph G, an edge $e = xy$ of G and a triple $(H; a, b)$ where a and b are distinct vertices of the graph H, by replacing the edge e by $(H; a, b)$, we mean taking the disjoint union of $G - e$ and H, and identifying x with a and y with b. For our purposes, it does not matter whether x is identified with a or with b.

For two positive integers p and q, a (p, q)-coloring of a graph G is a vertex coloring c of G with colors $\{0, 1, 2, \ldots, p - 1\}$ such that

$$(x, y) \in E(G) \Longrightarrow q \leq |c(x) - c(y)| \leq p - q.$$

The circular chromatic number is defined as

$$\chi_c(G) = \inf \{p/q : G \text{ is } (p, q)\text{-circular colorable}\}.$$

So for a positive integer k, a $(k, 1)$-coloring of a graph G is just an ordinary k-coloring of G. The circular chromatic number of a graph was introduced by Vince [3]

as "the star-chromatic number" in 1988. He proved that for every finite graph G, the infimum in the definition of the circular chromatic number is attained, so the circular chromatic number $\chi_c(G)$ is always rational. He also proved, among other things, that $\chi - 1 < \chi_c \leq \chi$, and $\chi_c(K_n) = n$.

For a (p,q)-coloring ϕ of a graph G, let $D_\phi(G)$ be the digraph with vertex set $V(G)$ and for every edge xy in G there is a directed edge (x,y) in $D_\phi(G)$, if $\phi(y) - \phi(x) = q \pmod{p}$.

Lemma A. [1] *For a graph G, $\chi_c(G) < p/q$ if and only if $D_c(G)$ is acyclic for some (p,q)-coloring c of G.*

The question determining which graphs have $\chi_c = \chi$ was raised by Vince [3]. It was shown by Guichard [1] that it is **NP**-hard to determine whether or not an arbitrary graph G satisfies $\chi_c(G) = \chi(G)$. In [5] X. Zhu surveyed many results on circular chromatic number and posed some open problems on this topic, among them the following problem ([5], Question 8.23).

Problem 1 *What is the complexity of determining whether or not $\chi_c(G) = \chi(G)$, if the chromatic number $\chi(G)$ is known?*

We answer this question, using the following theorem.

Theorem A. [2] *It is* **NP***-hard to determine whether a graph is 3-colorable or any coloring of it requires at least 5 colors.*

2 Complexity

Consider the graph K^- which is obtained from a copy of K_4 with vertices v_1, v_2, v_3, and v_4, by removing the edge $\{v_1, v_2\}$. In the following trivial lemma all equalities are in $\mathcal{Z}_4$.

Lemma 1 *In every $(4,1)$-coloring c of K^-,*

(a) if $c(v_1) = c(v_2)$, then $D_c(K^-)$ is acyclic and has no directed path between v_1 and v_2.

(b) if $c(v_1) - c(v_2) = 1$, then $D_c(K^-)$ is acyclic and has a directed path from v_1 to v_2.

(c) if $c(v_1) - c(v_2) = 2$, then $D_c(K^-)$ has a cycle.

Consider the graph H shown in Figure 1. One can easily check that $\chi(H) = 4$ and we have the following Lemma.

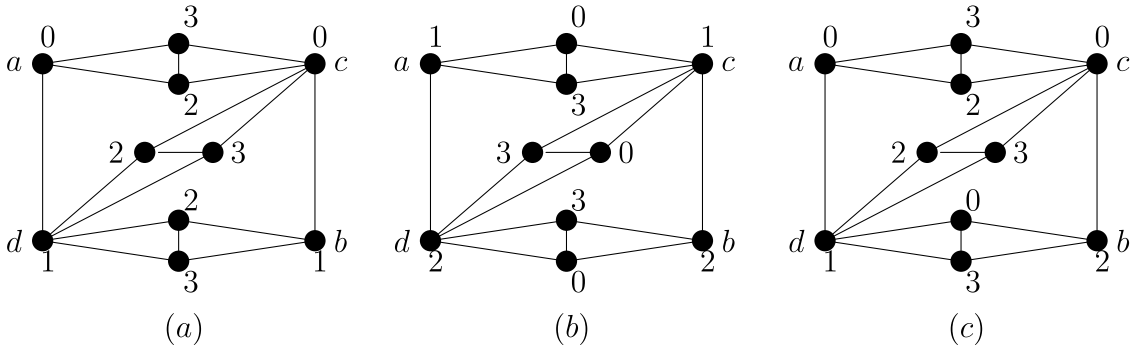

Figure 1: The graph H and its desired colorings

Lemma 2 *Consider the graph H shown in Figure 1.*

(a) For every $(4,1)$-coloring c of H, if $c(a) = c(b)$, then $D_c(H)$ has a cycle.

(b) For every $0 \leq x < y \leq 2$, there is a coloring c for H such that $c(a) = x$, $c(b) = y$, and $D_c(H)$ is acyclic and has no directed path from b to a.

Proof. (a) Without loss of generality assume that $c(a) = c(b) = 0$. For all cases except when $c(c) = c(d) = 1$ and $c(c) = c(d) = 3$, one can easily check by Lemma 1(c) that $D_c(H)$ has a cycle. Without loss of generality assume that $c(c) = c(d) = 1$. Now by Lemma 1(b) there are directed paths from d to b, b to c, c to a and a to d. Thus $D_c(H)$ has a cycle.

(b) Such colorings are given in Figures 1(a), 1(b), 1(c). ∎

Theorem 1 *Given a graph G and its chromatic number, the problem of determining whether or not $\chi_c(G) = \chi(G)$ is* **NP***-hard.*

Proof. For every graph G', we construct a graph G such that $\chi(G) = 4$, and if G' is 3-colorable, then $\chi_c(G) < 4$, and if G' is not 4-colorable, then $\chi_c(G) = 4$. Thus by Theorem A the result is proven.

Construct the graph G by replacing every edge of G' by $(H; a, b)$. Obviously, for every nontrivial graph G', $\chi(G) = 4$.

First suppose that G' is 3-colorable. So we can properly color the vertices of G' with 0, 1, and 2. Now by Lemma 2(b), this coloring can be expanded to a $(4,1)$-coloring c of G such that in $D_c(G)$ the copies of H are acyclic, and also for every two vertices u and v of G', there is no path from u to v in $D_c(G)$ if $c(u) > c(v)$. This implies that $D_c(G)$ is acyclic. So $\chi_c(G) < 4$.

Next suppose that G' is not 4-colorable. So in any $(4,1)$-coloring c of G there are two adjacent vertices u and v of G such that $c(u) = c(v)$. So by Lemma 2(a) for the copy of H which is between u and v there exists a cycle in $D_c(H)$. Hence $\chi_c(G) = 4$. ∎

Now we prove that it is **NP**-complete to verify that the difference between chromatic number and circular chromatic number of a given graph is greater than or

equal to $\frac{1}{k}$, when $k \geq 2$ is an arbitrary positive integer is **NP**-complete. Let K be a graph with vertex set $\{a, b, v_1, \ldots, v_{n-1}\}$ in which each v_i is adjacent to every other v_j, a is adjacent to $v_1, \ldots, v_{n-2}$, and b is adjacent to v_{n-1}.

Lemma 3 *For all integers $0 \leq x, y \leq kn-1$, K has a $(kn-1, k)$-coloring c with $c(a) = x$ and $c(b) = y$ if and only if $x \neq y$.*

Proof. If $x = y$, then a $(kn-1, k)$-coloring of K can be transformed to a $(kn-1, k)$-coloring of K_n by identifying a and b. And this is impossible because $\chi_c(K_n) = n$. If $x \neq y$ without loss of generality we can assume that $x = 0$ and $0 < y \leq \frac{kn-1}{2}$.

First suppose that $y \geq k$. In this case define a desired $(kn-1, k)$-coloring c by $c(a) = 0$, $c(b) = y$, $c(v_i) = ik$ for $1 \leq i \leq n-2$ and $c(v_{n-1}) = 0$.

Next suppose that $y < k$. In this case define a desired $(kn-1, k)$-coloring c by $c(a) = 0$, $c(b) = y$, $c(v_i) = ik$ for $1 \leq i \leq n-2$, and $c(v_{n-1}) = y - k$. ■

Theorem 2 *For all positive integers $k \geq 2$ and $n \geq 3$, the following problem is* **NP***-complete. A graph G is given where $\chi(G) = n$, and it is asked whether $\chi_c(G) \leq n - \frac{1}{k}$?*

Proof. Clearly, the problem is in **NP**. We reduce VERTEX COLORING to this problem. Consider a graph G' as an instance of VERTEX COLORING. It is asked whether the vertices of G' can be colored with $kn-1$ colors. We construct a new graph G with the property that $\chi_c(G) \leq n - \frac{1}{k}$ if and only if the vertices of G' can be colored with $kn-1$ colors.

Construct a graph G by replacing every edge uv of $G' \sqcup K_n$, the disjoint union of G' and a copy of K_n, by $(K; a, b)$. Obviously, $\chi(G) \leq n$. Since in every $(n-1)$-coloring of K the vertices a and b must have different colors, thus $\chi(G) = n$. We know that $\chi_c(G) \leq n - \frac{1}{k}$ if and only if there exists a $(kn-1, k)$-coloring c for G.

First suppose that $\chi(G') \leq kn-1$, and c is a $(kn-1)$-coloring of $G' \sqcup K_n$. For all copies of K in G, we have $c(a) \neq c(b)$. By Lemma 3, c can be extended to a $(kn-1, k)$-coloring of G. Thus $\chi_c(G) \leq n - \frac{1}{k}$.

Next suppose that $\chi(G') > kn-1$ and c is a $(kn-1, k)$-coloring of G. There exist two adjacent vertices u and v in G' such that $c(u) = c(v)$. But by Lemma 3, the copy of K between u and v has no $(kn-1, k)$-coloring. This is a contradiction. Thus $\chi_c(G) > n - \frac{1}{k}$.

■

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